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DYNAMICS OF BUBBLE OSCILLATION AND PRESSURE PERTURBATION IN AN EXTERNAL MEAN FLOW

Seung-Man Yang

Dept. of Chem. Eng., Korea Advanced Institute of Science and Technology, Taejon 305-701, Korea
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Abstract—The effects of the external mean flow on the bubble response to changes in the ambient pressure distribution are examined. The analysis of finite-amplitude shape oscillations of a constant-volume bubble in an *arbitrary* mean flow shows that a monopole pressure disturbance in the far-field can occur due only to the interaction between a disturbance flow associated with the shape oscillations and a special type of ambient flow. The result is *independent* of the degree of deformation and remains valid for any type of ambient pressure fluctuation that creates shape oscillations. Then, we specialize the problem to consider the nonlinear oscillation dynamics of a bubble in the presence of a *uniaxial straining* flow. In this case, the source of oscillations is a spatially inhomogeneous 'abrupt' change in the ambient pressure. The method of solution employed here is a domain perturbation in conjunction with a two timing analysis to examine small-amplitude oscillations of bubble shape relative to the *non-spherical* steady-state configuration in the ambient mean flow. The result shows that the ambient flow can interact directly with a mode of bubble deformation and produces a self-induced secularity that leads to a modification of the oscillation frequency. In addition, the disturbance pressure caused by shape oscillations exhibits at most a quadrupole character at large distances from the constant-volume bubble.

INTRODUCTION

This paper considers the nonlinear dynamic response of a bubble to a modulated acoustic pressure at the bubble surface, when the bubble is immersed in a steady mean flow at high Reynolds number. The nonlinear coupling between shape oscillations of a bubble and pressure variations in the surrounding fluid is responsible for a variety of important effects. These range from emulsification to acoustic noise generation in bubbly liquids [1, 2].

The free oscillation of drops and bubbles in a quiescent fluid has been studied extensively since the first mathematical model of *linear* droplet oscillations in vacuum due to Rayleigh [3]. The generalized linear solution to include the influence of a surrounding medium was given by Lamb [4]. The solution describes the instantaneous deformation of the bubble shape by an infinite series of the surface spherical harmonics, where each term corresponds to one independent natural oscillation mode. These linear results have been extended to include viscous effects [5, 6], and nonlinear oscillations of a liquid drop in a *quiescent* fluid have also been analyzed [7, 8]. In spite of the extensive literature on oscillating bubble in a quiescent fluid, however, relatively little has been done yet to determine how the oscillation dynamics are modified in the presence of a mean flow around the bubble. We are aware of only two papers in this direction, by Subramanyam [9] and Kang and Leal [10], who studied the oscillations of a *constant-volume* drop or bubble in another fluid at low values of Weber number. This provides one primary motivation for the present study of the oscillatory motion of a bubble in an undisturbed mean flow at infinity.

In the present paper, we consider the shape oscillation of a constant-volume bubble in response to an abrupt change in the ambient pressure in the presence of a mean motion relative to the bubble so that the steady-state bubble shape is *non-spherical*. The most closely existing analysis is due to Yang et al. [11] who considered small amplitude oscillations of an ideal gas bubble. In this case of a compressible bubble, there exist mode-mode interactions between the radial and shape oscillations and these interactions can exhibit resonance when the frequency of radial mode is matched with one of the shape modes. For example, when the frequency of the radial mode of oscillation is twice that of a normal mode of shape oscillation in a *quiescent* fluid, there is a *resonant* interaction, and the shape mode eventually loses all of its energy to the radial mode [12]. The volume oscillation produced by the radial mode yields 'monopole' emission of sound (The monopole sound corresponds to a disturbance pressure oscillation which decays in magnitude as 1/r, where r is the distance from the bubble). In addition, when the ideal gas bubble is immersed in the steady mean flow, a *self-induced* resonance is always present and leads to a decrease in oscillation frequency with increase in the degree of bubble deformation in the steady flow. On the other hand, for a constant-volume bubble, radial oscillations only exist as a consequence of the constant volume constraint, and are much smaller in magnitude than the corresponding shape oscillations. Furthermore, the resulting radial mode of oscillation of a constant-volume bubble cannot create monopole emission of sound in the absence of a mean flow, as pointed out by Prosperetti [13]. In his paper, Prosperetti noted that the pressure field caused by the motion of a constant-volume bubble has a dipole character if the center executes oscillatory motion, of the quadrupole type if the bubble undergoes oscillations from a prolate spheroidal shape to an oblate one, and of higher multipole types for other forms of oscillations[1]. Accordingly, the pressure perturbations are proportional to r^{-2}, r^{-3} or higher powers, which are to be compared with the slow decay proportional to r^{-1} associated with changes of volume. However, in the presence of a mean flow the oscillation dynamics and the resulting pressure disturbance will be modified due to the interaction with the flow. In particular, we will show that shape oscillations of any amplitude for a constant-volume bubble cannot create a monopole-like pressure disturbance at infinity, except in the presence of very special types of ambient flow.

We begin, in the next section, by formulating the governing equations and boundary conditions. Following this, we consider the nonlinear dynamics of an incompressible bubble in an *arbitrary* mean flow. In the first part, we show that a monopole emission of sound in the far-field can occur due only to the interaction between a disturbance flow associated with the shape oscillation and a special ambient flow. Then, we specialize the problem to consider the dynamic response of a constant-volume bubble to a spatially inhomogeneous 'abrupt' change in the ambient pressure, with the bubble immersed in a uniaxial axisymmetric straining flow or in a quiescent fluid. In this investigation, we employ the method of domain perturbation in conjunction with the multiple scaling analysis to analyze small amplitude oscillations of bubble shape relative to the non-spherical steady-state configuration in the ambient mean flow.

1) N-th order *multipole* pressure field such as dipole and quadrupole denotes a pressure perturbation which decays in magnitude as r^{-N} ($N \geq 2$).

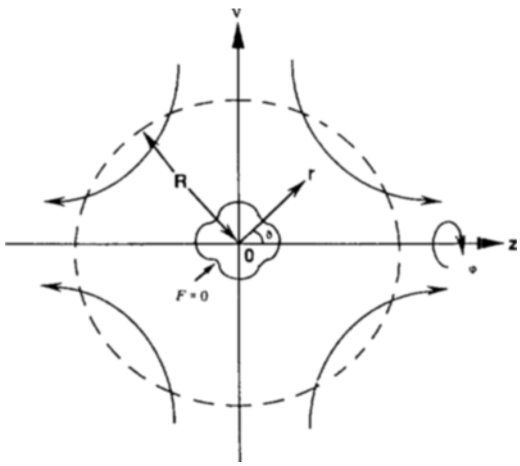

Fig. 1. Definition of the spherical coordinate system for an oscillating bubble. The spherical surface of radius R encloses the bubble. The z-axis of a cylindrical coordinate system (z, v, φ) is directed along the axis of symmetry and $r=\sqrt{z^2+v^2}$. The bubble surface is defined by F=0.

BASIC EQUATIONS

We consider an incompressible bubble of volume $4\pi a^3/3$ which is undergoing oscillations of shape in the presence of an undisturbed flow of an inviscid fluid with density ρ as sketched in Fig. 1. The surface of the bubble is assumed to be clean and mobile, and characterized completely by a constant interfacial tension σ. In the absence of pressure fluctuation or external mean flow, the pressure is uniform with magnitude p^o and the bubble is spherical with an equilibrium radius a. Furthermore, we neglect all effects of gravity including the hydrostatic pressure variation in the fluid. In order to non-dimensionalize the governing equations for bubble oscillations, we introduce characteristic velocity (u_c), length (l_c), and time (t_c) scales. Since the dynamic pressure difference is balanced with the surface tension at the bubble surface at all times, the most appropriate choice is

$$u_c=\sqrt{\frac{\sigma}{\rho a}},\ l_c=a,\ t_c=\sqrt{\frac{\rho a^3}{\sigma}}. \tag{1}$$

Then the governing equations appropriate to the problem described above can be expressed in terms of the velocity potential Φ and the pressure field p

$$\nabla^2\Phi=0 \tag{2}$$

$$\Phi_t+\frac{1}{2}(\nabla\Phi\cdot\nabla\Phi)+p=p^o \tag{3}$$

where the subscript denotes the differential variable.

We now specify the bubble surface in terms of spherical polar coordinates(r, θ, φ), by $F\equiv r-1-f(\theta, \varphi, t)=0$. In this study, we assume that the source of the oscillations in bubble shape is a time-dependent spatially inhomogeneous *acoustic* pressure $A(\theta, \varphi, t)$ at the bubble surface. This type of surface pressure modulation can be produced experimentally via ultrasonic acoustic wave fields, without generating any net motion of the suspending fluid [14, 15]. To describe the bubble oscillations in response to the modulated acoustic pressure on the interface, the disturbance flow field must be determined with the acoustic radiation pressure incorporated into the boundary conditions at the bubble surface [16, 17]. On the bubble surface the kinematic condition and the normal stress jump condition must be satisfied:

$$-F_t=\nabla\Phi\cdot\nabla F \tag{4}$$

$$\tilde{p}-p^o-A(\theta, \varphi, t)+\Phi_t+\frac{1}{2}(\nabla\Phi\cdot\nabla\Phi)=\nabla\cdot\left(\frac{\nabla F}{|\nabla F|}\right) \tag{5}$$

where $\tilde{p}$ is an unknown time-dependent pressure inside the bubble which must be determined to satisfy the constraint of volume conservation. The far field boundary condition corresponding to the presence of an undisturbed mean flow can be expressed as:

$$\Phi\rightarrow\Phi^u \text{ as } r\rightarrow\infty \tag{6}$$

where Φ^u represents the velocity potential for the undisturbed flow.

In addition to the differential equations and boundary conditions (2)-(6), the solution for bubble shape must satisfy the constant volume constraint, i.e.,

$$\langle(1+f)^3\rangle\equiv\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}(1+f)^3\sin\theta d\theta d\varphi=1 \tag{7}$$

in which $\langle(\cdot)\rangle$ denotes the spherical surface average of the quantity $(\cdot)$.

FINITE-AMPLITUDE SHAPE OSCILLATIONS

In the presence of a mean flow at infinity, the disturbance pressure field for an incompressible bubble is a consequence of the interaction between the disturbance flow produced by shape oscillations and the undisturbed mean flow. The interaction between changes of bubble shape and mean motion of the surrounding fluid can be determined in a general frame work where it is not necessary to limit the analysis to the small deformation limit. Suppose Φ^s is the velocity potential of the steady-state solution for a constant-volume bubble in a mean flow as defined by the velocity potential Φ^u. Suppose also that $\tilde{\phi}$ is the disturbance potential due to shape oscillations which are caused by the acoustic pressure modulation. These velocity potentials can be expressed in terms of the spherical surface harmonics $S_n(\theta, \varphi)$ of order $n(n=1, 2,\cdots)$ in the form:

$$\Phi^s=\sum_{j=1}^{k} B_j r^j S_j(\theta, \varphi)+\sum_{j=0}^{\infty} C_j\frac{1}{r^{j+1}}S_j(\theta, \varphi) \tag{9}$$

$$\tilde{\phi}=\sum_{j=0}^{\infty} D_j(t)\frac{1}{r^{j-1}}S_j(\theta, \varphi) \tag{10}$$

where B_j, C_j are constants and $D_j(t)$ is a function of time t only. The sum of the terms with coefficients B_j in (9) corresponds to the velocity potential Φ^u for the undisturbed mean flow. Note that the form (10) does not assume that the deformations of bubble shape are necessarily small.

To determine the pressure field, we utilize the equation of motion (3). The disturbance pressure at infinity p_d^∞ which comes from the shape oscillations is then defined by

$$p_d^\infty=p^\infty-\left\{p^o-\frac{1}{2}(\nabla\Phi^s)^2\right\}. \tag{11}$$

As a first step, we show that $C_0=D_0(t)=0$ for a constant-volume bubble. At *steady-state*, the volume flux across any circumscribed spherical surface should be zero for an incompressible fluid, i.e.,

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(\frac{\partial\Phi^s}{\partial r}\right)_{r=R}\sin\theta d\theta d\varphi=0 \tag{12}$$

where R is the radius of an arbitrary spherical surface, which is centered at the origin and encloses the bubble. Since

$$\int_0^{2\pi}\int_0^{\pi}S_j(\theta, \varphi)\sin\theta d\theta d\varphi=0 \text{ for } j\neq0,$$

it follows from (9) and (12) that

$$C_0=0. \tag{13}$$

For the unsteady case where the bubble executes shape oscillations, the net volume flux across the same spherical surface is still equal to zero. Thus, it follows from (10) that

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left[\frac{\partial(\Phi^s+\tilde{\Phi})}{\partial r}\right]_{r=R}\sin\theta d\theta d\varphi = D_0(t) = 0 \qquad (14)$$

and we see from (10) and (14) that shape oscillations of a constant-volume bubble cannot generate a monopole-like (i.e., source-like) disturbance of the *velocity potential.* This result is clearly *independent* of the degree of deformation and type of flow, and is solely due to the constant-volume constraint.

It thus follows from the equation of motion (3) that the only possible source of a monopole contribution to the pressure field is the term

$$\nabla\Phi^s\cdot\nabla\tilde{\Phi}. \qquad (15)$$

Further, for a constant-volume bubble, we see from (10) and (14) that

$$\tilde{\Phi} = D_2(t)\frac{1}{r^3}S_2(\theta, \varphi) + O(r^{-4}) \text{ as } r\to\infty. \qquad (16)$$

Here, we exclude the term involving $S_1(\theta, \varphi)$ which corresponds to a simple translation of the sphere. Thus, if the bubble oscillation is to create a monopole pressure distribution, the potential Φ^s of the *steady-state* undisturbed external flow must vary at large distances from the bubble as r^4. Indeed, if

$$\Phi^\infty = B_4r^4S_4(\theta, \varphi) + O(r^3) \text{ as } r\to\infty \qquad (17)$$

the bubble oscillation will create a *monopole* pressure distribution. The monopole pressure can be determined simply from (11), (16) and (17). The result is

$$p_d^\infty \sim B_4D_2(t)\left(12S_2S_4 - \frac{\partial S_2}{\partial\theta}\frac{\partial S_4}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial S_2}{\partial\varphi}\frac{\partial S_4}{\partial\varphi}\right)\frac{1}{r} \qquad (18)$$

as $r\to\infty$.

This monopole pressure fluctuation is due solely to the interaction between the particular external mean flow corresponding to (17) and the disturbance field generated by the shape oscillations of a constant-volume bubble. A monopole-like pressure disturbance is *not* possible for an incompressible bubble in either a quiescent fluid, or in the presence of a linear mean flow. The surprising feature here is that there is a form for the external mean flow which does produce a monopole-like contribution to the far field pressure distribution. However, the practical significance of this result is limited since a flow of the required form is difficult to achieve. For a uniform flow, a quiescent fluid, or even flows which depend linearly or quadratically on the spatial position, our analysis has demonstrated that a monopole-like dependence of the pressure disturbance cannot be created by shape oscillations of an incompressible bubble. However, when the bubble center executes oscillatory motion, we must include the $S_1(\theta, \varphi)$ mode in (16). In this case, a constant-volume bubble can emit a monopole pressure perturbation in the presence of a *quadratic* mean flow [i.e., $\Phi^\infty = B_3r^3S_3(\theta, \varphi) + O(r^2)$]. It is noteworthy that the general features discussed above are *independent* of the degree of bubble deformation, and remain valid for any type of ambient pressure fluctuation that creates shape oscillations.

SMALL-AMPLITUDE OSCILLATIONS IN A UNIAXIAL STRAINING FLOW

In this section, we study shape oscillations of a constant-volume bubble, when the fluid undergoes a uniaxial straining flow. We are especially interested in the influence of the steady deformation due to the ambient flow, and in the consequence of the radial mode oscillations that are required in the presence of shape oscillations to maintain the bubble volume constant. The far-field velocity potential Φ^u for the presence of the undisturbed straining flow with the principal strain rate E can be expressed in terms of the second order Legendre polynomial $P_2(\zeta)$:

$$\Phi^u \equiv \varepsilon^{1/2}\frac{r^2}{2}P_2(\zeta) \qquad (19)$$

where ε is the Weber number, defined by $\varepsilon\equiv\rho(Ea)^2a/\sigma$ and $\zeta\equiv\cos\theta$ with the angle ν measured from the axis of symmetry.

We now specialize the problem to consider the small-amplitude oscillations of a constant-volume bubble caused by an *abrupt* change in the pressure on the bubble surface at some initial instant, when the bubble is in a uniaxial axisymmetric straining flow. Without loss of generality, we assume that the applied pressure modulation A is axisymmetric, i.e., $A = A(\theta, t)$, so that the deformed shape of the bubble remains axisymmetric. The obvious requirements for the limit of small deformation where the shape is nearly spherical are $\varepsilon \ll O(1)$ and $A(\theta, t) \ll O(1)$. To simplify the analysis, it is convenient to use the method of domain perturbations for the limit $A(\theta, t) = O(\varepsilon)$ to transform the kinematic and the normal stress conditions at the bubble surface to equivalent conditions applied at r=

1. The method is well-known and we simply quote the results for the kinematic condition [11], which becomes

$$f_t-\Phi_r=f\Phi_{rr}-(1-\zeta^2)\ (f_\zeta\Phi_\zeta+2ff_\zeta\Phi_\zeta-ff_\zeta\Phi_{r\zeta})$$
$$-\frac{1}{2}f^2\Phi_{rrr}+O(f^3)\ \text{at}\ r=1 \quad (20)$$

and the normal stress condition, which takes the form

$$\Phi_t+2f+\nabla_s^2f+G(t)=A(\theta,\ t)-f\Phi_{tr}-\frac{1}{2}(\nabla\Phi)^2$$
$$+2f(f+\nabla_s^2f)-f(\nabla\Phi)\cdot(\nabla\Phi)_r-\frac{f^2}{2}\Phi_{trr}-f^2(2f+3\nabla_s^2f)$$
$$+(1-\zeta^2)(f_\zeta)^2\left(\frac{3}{2}\nabla_s^2f+\zeta f_\zeta\right)+O(f^4)\ \text{at}\ r=1 \quad (21)$$

in which we define the internal bubble pressure as

$$\tilde{p}=\tilde{p}^0+G(t) \quad (22)$$

and we utilize the equilibrium condition $\tilde{p}^o-p^o=2$. Here, ∇_s^2 denotes the surface Laplacian:

$$\nabla_s^2\equiv\frac{\partial}{\partial\zeta}\left\{(1-\zeta^2)\frac{\partial}{\partial\zeta}\right\}$$

The unknown time-dependent constant G(t) represents the pressure change inside the bubble due to the *shape* oscillation.

In the asymptotic limit outlined above, it is usual to expand the solution in the form of a perturbation expansion for small $\varepsilon\ll1$, i.e.

$$\Phi=\varepsilon^{1/2}\Phi_0+\varepsilon\Phi_1+\varepsilon^{3/2}\Phi_2+\cdots, \quad (23)$$

$$f=\varepsilon f_1+\varepsilon^{3/2}f_2+\varepsilon^2f_3+\cdots, \quad (24)$$

and

$$G(t)=\varepsilon G_1+\varepsilon^{3/2}G_2+\varepsilon^2G_3\cdots. \quad (25)$$

We expand the bubble shape function $f_j(\zeta,\ t)$ at each order in $\varepsilon^{1/2}$ as an infinite series of the Legendre polynomials $P_n(\zeta)$ of order n:

$$f_j(\zeta,\ t)=\sum_{n=0}^{\infty}a_{j,n}(t)P_n(\zeta),\ j=1,\ 2\cdots \quad (26)$$

and the velocity potential Φ_j at each order in $\varepsilon^{1/2}$ in terms of solid spherical harmonics: for j=0,

$$\Phi_0=\Phi^u+\sum_{n=0}^{\infty}b_{0,n}\frac{P_n(\zeta)}{r^{n+1}} \quad (27)$$

and for j≥1,

$$\Phi_j=\sum_{n=0}^{\infty}b_{j,n}(t)\frac{P_n(\zeta)}{r^{n+1}}\ . \quad (28)$$

All that remains is to determine the time-dependent coefficient functions $a_{j,n}(t)$ and $b_{j,n}(t)$ which satisfy the kinematic and the normal stress conditions (20), and (21) at the bubble surface.

The steady-state problem with $A(\theta,\ t)=0$ was studied by Kang and Leal [10], and we simply quote their results here. For the present study, we need the velocity potential Φ^s at the steady state up to $O(\varepsilon^{3/2})$ which has the form:

$$\Phi^s=\varepsilon^{1/2}\Phi_0^s+\varepsilon^{3/2}\Phi_2^s+O(\varepsilon^2)\ \text{with}\ \Phi_1^s=0 \quad (29)$$

where

$$\Phi_0^s=\left(\frac{1}{2}r^2+\frac{1}{3}r^{-3}\right)P_2(\zeta) \quad (30)$$

and

$$\Phi_2^s=\left[\frac{275}{9072}\frac{P_2(\zeta)}{r^3}+\frac{325}{5544}\frac{P_4(\zeta)}{r^5}-\frac{125}{4158}\frac{P_6(\zeta)}{r^7}\right]. \quad (31)$$

Here, the superscript s will be used hereafter to denote the steady-state solution. For the present purpose, the shape function f^s at the steady state is required up to $O(\varepsilon)$ and can be expressed as:

$$f^s=\varepsilon\left[\frac{25}{336}P_2(\zeta)-\frac{5}{126}P_4(\zeta)\right]+O(\varepsilon^2). \quad (32)$$

Let us first consider the bubble response in the uniaxial straining flow to an *impulse* in the pressure at the bubble surface. In this case, the function A(ν, t) can be expressed in terms of the Legendre polynomials $P_n(\zeta)$ for an axisymmetric mode:

$$A(\theta,\ t)=\varepsilon\sum_n A_n\delta(t)P_n(\zeta) \quad (33)$$

where εA_n is the impulse magnitude of the $P_n(\zeta)$ mode and δ(t) is the Dirac-delta function. The impulsive change of the pressure at O(ε) induces an initial O(ε) oscillation of the bubble shape. The equations of motion and boundary conditions for the linear approximation at O(ε) admit solutions in the form of normal modes:

<u>O(ε) Solution:</u>

For n≥2,

$$a_{1,n}(t)=-\frac{(n+1)A_n}{\omega_n}\sin\omega_nt+a_{1,n}^s \quad (34)$$

$$b_{1,n}(t)=A_n\cos\omega_nt. \quad (35)$$

For n=0 and 1,

$$a_{1,n}(t)=b_{1,n}(t)=0. \quad (36)$$

Here, ω_n is the natural frequency of the $P_n(\zeta)$ mode ($n\geq 2$) and is defined by

$$\omega_n \equiv \sqrt{(n-1)(n+1)(n+2)}. \tag{37}$$

Thus, the problem can be viewed as an initial value problem with

$$a_{j,n}(0)=0;\ \frac{\partial}{\partial t} a_{1,n}(0)=-(n+1)A_n,$$
$$\frac{\partial}{\partial t} a_{j-1,n}(0)=0 \quad (j\geq 1) \tag{38}$$

for $n\geq 2$. The existence of a radial mode of oscillation for an incompressible bubble is solely a consequence of the fact that the $O(\varepsilon)$ *shape* oscillation (34) produces a volume change at $O(\varepsilon^2)$, and this must be compensated by an $O(\varepsilon^2)$ contribution from the radial mode to maintain constant volume.

To determine the *shape* oscillations at higher order, we adopt the asymptotic expansion scheme (23)-(25) together with the general solution forms (26)-(28), and derive governing differential equations for the coefficient $a_{j,n}(t)$ and $b_{j,n}(t)$. This process is straightforward. However, an important feature of the analysis is that the governing differential equations for $a_{j,n}(t)$ ($n\geq 2$) turn out to contain *secular* terms of *self-induced* type, which are present whenever the coefficients $a^s_{1,n}$ for the steady-state shape are non-zero (i.e., whenever the bubble is deformed at steady state). Since we expect bounded solutions for the various shape modes [recall that the pressure change is small, of $O(\varepsilon)$], it is evident that the form of the solutions (34)-(36) are overly simplified. In particular, experience with similar problems indicates the necessity of allowing a slow variation of the *amplitude* or *phase* of each mode on a time scale

$$\tau=\varepsilon t \tag{39}$$

to be superimposed on the higher frequency oscillation that is at (or near) the natural frequency ω_n. The asymptotic analysis is then carried out via a standard two-timing procedure.

With this change, the $O(\varepsilon)$-solution is now expressed in terms of normal modes, (26)-(28), in which the coefficient functions take the form

$$a_{1,n}(t,\ \tau)=\frac{1}{2}[\beta_{1,n}(\tau)e^{i\omega_n t}+a^s_{1,n}]+\text{c.c.} \tag{40}$$

$$b_{1,n}(t,\ \tau)=-\frac{i\omega_n}{2(n+1)}\beta_{1,n}(\tau)e^{i\omega_n t}+\text{c.c.} \tag{41}$$

with

$$\beta_{1,n}(0)=\frac{A_n(n+1)i}{\omega_n} \tag{42}$$

for $n\geq 2$. Here, $\beta_{1,n}(\tau)$ is a complex function (i.e., $i=\sqrt{-1}$) and c.c. denotes the complex conjugate of the preceding term(s). The slowly varying function $\beta_{1,n}(\tau)$ is chosen in such a way that the *self-induced* secular terms at $O(\varepsilon^2)$ are cancelled in the governing equation for $a_{3,n}$ so that the solution for $a_{3,n}$ remains bounded for all t. This procedure leads to a differential equation for $\beta_{1,n}(\tau)$.

In the presence of the uniaxial straining flow, the second term in the expansion for f and Φ occurs at $O(\varepsilon^{3/2})$. Governing equations for the P_2, and P_4 modes of oscillation at $O(\varepsilon^{3/2})$, can be obtained from the general Eqs. (20) and (21) and the forms of the $O(\varepsilon)$ solutions above. However, it is not necessary to report all of the details, except to introduce the solution at $O(\varepsilon^{3/2})$ and this solution then will be used to derive the governing equations at $O(\varepsilon^2)$.

<u>$O(\varepsilon^{3/2})$ Solution</u>

<u>P_2 Mode</u>

$$a_{2,2}(t,\tau)=\frac{125}{1638}\left[A_4e^{i\omega_2 t}+\frac{i\omega_4}{5}\beta_{1,4}(\tau)e^{i\omega_4 t}\right]+\text{c.c.} \tag{43}$$

$$b_{2,2}(t,\tau)=\frac{b^s_{2,2}}{2}+\frac{5}{42}\left[\beta_{1,2}(\tau)-\frac{25i\omega_2A_4}{117}\right]e^{i\omega_2 t}$$
$$+\frac{35}{117}\beta_{1,4}(\tau)e^{i\omega_4 t}+\text{c.c.} \tag{44}$$

<u>P_4 Mode</u>

$$a_{2,4}(t,\tau)=\frac{75}{546}\left[A_2e^{i\omega_4 t}+\frac{i\omega_2}{3}\beta_{1,2}(\tau)e^{i\omega_2 t}\right]+\text{c.c.} \tag{45}$$

$$b_{2,4}(t,\tau)=\frac{b^s_{2,4}}{2}+\frac{7}{13}\beta_{1,2}(\tau)e^{i\omega_2 t}$$
$$+\frac{5}{77}\left[\beta_{1,4}(\tau)-\frac{11i\omega_4A_2}{26}\right]e^{i\omega_4 t}+\text{c.c.} \tag{46}$$

Obviously, the radial mode does *not* appear at $O(\varepsilon^{3/2})$. A constant-volume bubble will execute the radial-mode oscillation at $O(\varepsilon^2)$ to compensate the volume change associated with the *shape* oscillations at $O(\varepsilon)$.

The governing equations at $O(\varepsilon^2)$ are again obtained from the kinematic and the dynamic boundary conditions (20) and (21), and now depend on the forms of the solutions for both $a_{j,n}$, $b_{j,n}$ ($j=1, 2$). For brevity, we show explicitly, only the equations for $a_{3,0}$, $a_{3,2}$, and $a_{3,4}$.

<u>$O(\varepsilon^2)$ Problem</u>

<u>Radial Mode</u>

$$a_{3,0}(t,\tau) = -\sum_{n\geq 2} [a_{1,n}(t,\tau)]^2 \frac{1}{2n+1} \tag{47}$$

P₂ Mode

$$\left(\frac{\partial^2}{\partial t^2} + \omega_2{}^2\right) a_{3,2}(t,\tau) = \left(-i\omega_2 \frac{\partial}{\partial \tau} + \frac{3040}{819}\right) \beta_{1,2}(\tau) e^{i\omega_2 t} + \text{c.c.} + \text{n.s.t.} \tag{48}$$

P₄ Mode

$$\left(\frac{\partial^2}{\partial t^2} + \omega_4{}^2\right) a_{3,4}(t,\tau) = \left(-i\omega_4 \frac{\partial}{\partial \tau} + \frac{25}{4}\,\frac{34649}{48279}\right) \beta_{1,4}(\tau) e^{i\omega_4 t} + \text{c.c.} + \text{n.s.t.} \tag{49}$$

where n.s.t. denotes the non-secular terms.

As noted earlier, there are self-induced *secular* terms at $O(\varepsilon^2)$, which are everpresent due to the non-spherical steady-state shape in the straining flow. To obtain bounded solutions at $O(\varepsilon)$, the slowly varying coefficients $\beta_{1,n}(\tau)$ in the $O(\varepsilon)$ solutions must be chosen to eliminate the self-induced secular terms in (48) and (49). Thus, the solution which satisfies the initial condition (42) for an impulsive change of the ambient pressure, as defined in (33), is

$$\beta_{1,n}(\tau) = i\frac{A_n(n+1)}{\omega_n} \exp(-i\omega_n Q_n \tau) \tag{50}$$

where Q_n is a constant that depends on n, e.g., $Q_2 = 0.3093$, $Q_4 = 0.0498$, etc. It follows from (50) that the complete solution at $O(\varepsilon)$ for the impulsive pressure response in terms of the coefficients $a_{1,n}$ and $b_{1,n}$ is

P_n (n≥2) Mode

$$a_{1,n}(t) = a_{1,n}^s - \frac{A_n(n+1)}{\omega_n} \sin[\omega_n(1-Q_n\varepsilon)t] \tag{51}$$

$$b_{1,n}(t) = A_n \cos[\omega_n(1-Q_n\varepsilon)t] \tag{52}$$

where $a_{1,n}^s$ denotes the steady-state shape and is given by $a_{1,2}^s = (25/336)$, $a_{1,4}^s = -(5/126)$, and others = 0.

Among the most interesting and important results, evident from (51) and (52) is that the frequency of shape oscillation is *modified* by the presence of the extensional flow. This frequency modification is due solely to the *non-spherical* steady-state shape and is not present in the absence of the external flow. The fact that the frequency of the shape oscillation decreases in a straining flow has an important fundamental significance, because at the critical Weber number ε_c, where the square of the frequency of oscillation becomes zero, eigenvalues for the coefficients functions $a_{1,n}$ change from pure imaginary to real. The critical Weber number where this occurs will thus correspond exactly to a limit point for existence of the corresponding steady-state solution. An asymptotic prediction of the critical Weber number can therefore be obtained from the present solution (51), namely

$$\varepsilon_c = \min\left(\frac{1}{2Q_n}\right), \quad Q_n > 0 \quad (n \geq 2)$$

or

$$\varepsilon_c = 1.616.$$

Kang and Leal [10] studied small amplitude oscillations of a constant-volume bubble and showed that the steady-state solution for the bubble shape suggests the existence of a limit point at a critical Weber number, beyond which no solution exists on the steady-state solution branch which includes the spherical equilibrium state in the absence of flow (e.g., the critical value of 1.73 was estimated from the third-order solution). The estimate of the critical value predicted here at which the stability of the solution branch is exchanged is much closer to the numerical value of 1.38 estimated by Miksis [18] and Ryskin and Leal [19]. In addition, Kang and Leal [10] also determined the eigenvalues directly by considering mode-mode interactions between the first two even orders without inclusion of higher-order modes. From these eigenvalues the critical Weber number was found to be 1.613 which is almost identical to the present result.

The radial mode which must be added to conserve volume at $O(\varepsilon^2)$ is

Radial Mode

$$a_{1,0}(t) = a_{2,0}(t) = 0, \quad a_{3,0}(t) = -\sum_{n\geq 2} [a_{1,n}(t)]^2 \frac{1}{2n+1} \tag{53}$$

$$b_{j,0}(t) = 0 \quad (j \geq 1). \tag{54}$$

The significance of the solutions (53) and (54) for the radial mode is twofold. First, the frequency of this radial mode for a constant-volume bubble will be quite different from the natural frequency ω_0 for radial oscillations of an ideal gas bubble. In a previous study, Longuet-Higgins considered small-amplitude oscillations of an *ideal gas* bubble caused by relaxation from an initially distorted shape of $O(\varepsilon)$ in a quiescent fluid [20]. He found that at second order, $O(\varepsilon^2)$, the distortion modes create a radial mode of oscillation with the natural frequency

$$\omega_o \equiv \sqrt{3\gamma\, \bar{p}^o - 2} \tag{55}$$

where the parameter γ depends on the thermodynamic nature of the bubble oscillation. For isothermal oscillation γ is unity and for an adiabatic process γ is the

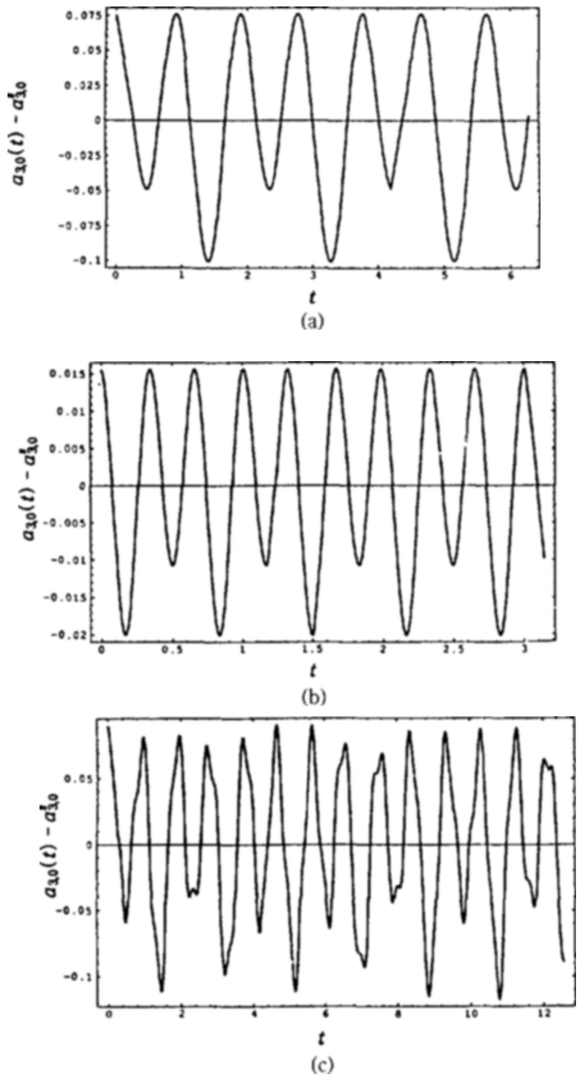

Fig. 2. Oscillation of the amplitude function $a_{3,0}(t)$ around its steady-state value $a_{3,0}^{s}$ for three modes of the pressure impulse: (a) $A_2=1$, $A_n=0$, ($n\geq 3$), $\varepsilon=0.1$; (b) $A_4=1$ (other$=0$), $\varepsilon=0.1$; (c) $A_2=A_4=1$ (others $=0$), $\varepsilon=0.1$.

ratio of the specific heat capacities ($\gamma=C_p/C_v$). Otherwise, γ ranges between these two limiting values. For a constant-volume bubble, however, the predicted frequency consists of a superposition of freqencies for the shape oscillations, $a_{1,n}(t)$ ($n\geq 2$). In Figs. 2a-2c, the radial mode oscillations, i.e., $a_{3,0}$ vs. t are illustrated for three different modes of the pressure impulse. As noted, the oscillation pattern of radial mode is clearly dependent upon the mode of impulse in the ambient pressure, which is quite different from the result for a compressible bubble.

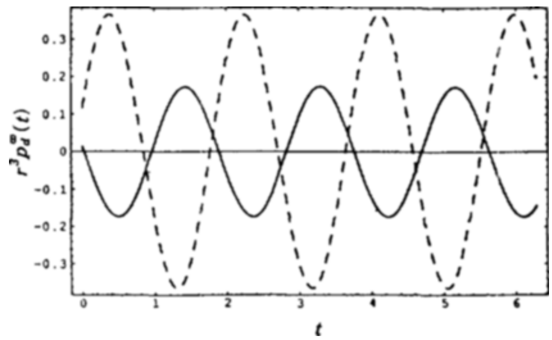

Fig. 3. Fluctuating pressure p_d^{∞} at large distances from the oscillating bubble for $A_2=1$ (others$=0$), $\varepsilon=0.1$. The dashed line is for the pole ($\theta=0$) and the solid line for the equator ($\theta=\pi/2$).

The second point of significance of the solutions (53) and (54) is that, for an incompressible bubble, the radial mode oscillation cannot generate a pressure disturbance that resembles a monopole, in the sense that it decays like r^{-1}. As we noted earlier, the radial oscillation of an ideal gas bubble emits the monopole disturbance pressure. However, this monopole pressure contribution is due to changes in the bubble volume [13, 20]. For a constant volume bubble, it is still necessary to include radial mode contributions to the bubble's motion at $O(\varepsilon^2)$, but these exist only to *cancel* volume changes associated with the change in shape at $O(\varepsilon)$, and the disturbance flow-field does *not* contain a source-like disturbance at *any* order in ε, i.e., $b_{j,0}(t)=0$ ($j\geq 1$). Since the major contribution to the disturbance pressure comes from the $P_2(\zeta)$-mode oscillation, it is sufficient to consider the pressure impulse with the same mode [i.e., $A=A_2P_2(\zeta)\,\delta(t)$]. Indeed, the disturbance pressure due to the shape oscillation behaves like r^{-3} at large distances from the constant-volume bubble:

$$p_d^{\infty}=\varepsilon A_2\omega_2\sin\{\omega_2(1-\varepsilon Q_2)t\}\frac{P_2(\zeta)}{r^3}+\varepsilon^{3/2}A_2 \cos\{\omega^2(1-\varepsilon Q_2)t\}\frac{8P_2(\zeta)+18P_4(\zeta)}{7r^3}+O(r^{-4}) \quad (56)$$

The fluctuating pressure at large distances is plotted as a function of time t in Fig. 3 for two different polar angles $\theta=0$, $\pi/2$. It can be easily seen that the amplitude of oscillating pressure at the pole ($\theta=0$) is larger than that on the equator ($\theta=\pi/2$). This is due to the fact that the extensional flow makes the amplitude of bubble-shape distortion large at the pole compared

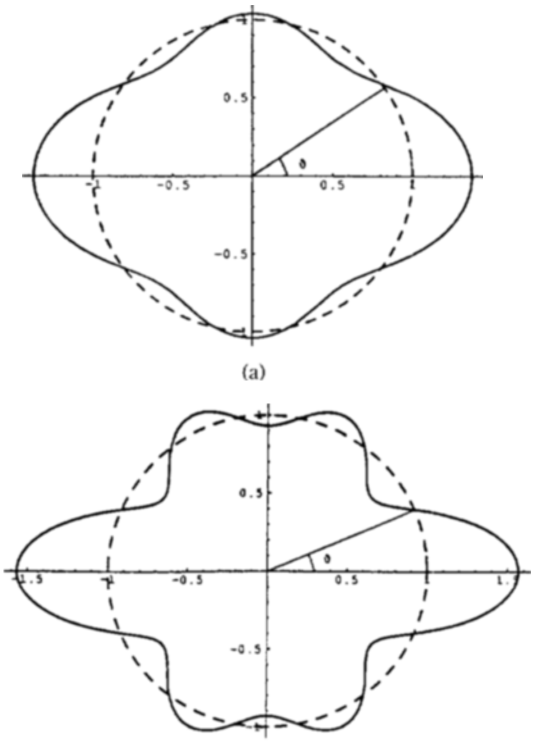

Fig. 4. Initial pressure distribution $(p_d^{\infty})_{t=0}$ at large distances from the bubble for two different modes of the pressure impulse with $\varepsilon=0.1$: (a) $A_2=1$ (others $=0$); (b) $A_4=1$ (others$=0$). The dashed circle represents the steady pressure distribution in the presence of the extensional flow.

with that on the equator. Further, the pressure oscillations at the two locations are out of phase with an identical frequency.

An important objective of the present study is the pressure disturbance at large distances from the bubble owing to the interaction between the shape oscillation and the extensional flow. Hence, we consider the origin of each term in (56) in detail. First, the $O(\varepsilon)$ term comes from the $P_2(\zeta)$-mode of shape oscillation at the same order and would be present with $Q_2=0$ in a quiescent fluid. On the other hand, the $O(\varepsilon^{3/2})$ term arises from the interaction between the shape oscillations and the external mean flow and would not be present in the absence of the straining flow, i.e., the $O(\varepsilon^{3/2})$ contribution to the pressure disturbance is associated with the Bernoulli term $(\nabla\Phi_0)\cdot(\nabla\Phi_1)$ and $\partial\Phi_2/\partial t$ in the equation of motions. The two contribution to the pressure perturbations are anti-phase. Thus, the initial disturbance pressure at large distances is due only to the influence of the straining flow and given by

$$(p_d^{\infty})_{t=0} \approx \varepsilon^{3/2}\,\frac{8P_2(\zeta)+18P_4(\zeta)}{7r^3}. \tag{57}$$

In Fig. 4a and 4b, the initial distributions of disturbance pressure at large distances are plotted as a function of the polar angle θ for the P_2 and P_4 mode of pressure impulses, respectively. It can be noted that the disturbance pressure exhibits higher order mode distribution than the mode of the impulse. This is due to the mode-mode interactions, or the evolution of a low order mode oscillation to higher order mode. The proportional amplification is therefore

$$\frac{p_d^{\infty}}{(p_d^{\infty})_{t=0}} = O(\varepsilon^{-1/2}) \tag{58}$$

which is large in the asymptotic limit $\varepsilon \ll 1$.

By taking the spherical surface average of the dynamic boundary condition (21), we can also determine the distrubance pressure inside the bubble

$$G(t) = -\varepsilon\frac{5}{12} - \varepsilon^{3/2}A_2\cos\{\omega_2(1-\varepsilon Q_2)t\} + O(\varepsilon^2) \tag{59}$$

where the factor $-5/12$ at $O(\varepsilon)$ is required to balance the ambient pressure change induced by the extensional flow. Thus, the internal pressure fluctuates with the same frequency as the $P_2(\zeta)$ mode of deformation. It is worth pointing out, for this case of a constant-volume bubble, that a radially symmetric pressure pulse (i.e., $A_0\neq0$, $A_n=0$, $n\geq2$) will only induce a disturbance in the internal pressure and *cannot* generate any shape oscillation. On the other hand, an axisymmetric but non-isotropic pressure disturbance (i.e., $A_n\neq0$) leads to both radial and shape oscillations in addition to the internal pressure disturbance.

Now, we may mention briefly the response of the constant-volume bubble to an impulsive change of pressure when the fluid is not undergoing any steady motion. The analysis is analogous to that considered above, and in most respects the behavior of the bubble is similar. Specifically, a radial mode of oscillation exists at $O(\varepsilon^2)$ to satisfy the constant-volume constraint, but the equivalent source strength $-b_{j,0}(t)$ generated by the bubble oscillation is identically zero. The chief difference between this limiting case and the bubble in an external flow is that there is *no* secular behavior at $O(\varepsilon^2)$. Hence, there is no frequency modification in this case and leading order nonzero solution for the radial and deformation modes of oscillation is simply

$$a_{1,n}(t) = -\frac{A_n(n+1)}{\omega_n}\sin\omega_n t,\ (n\geq 2) \quad (60)$$

$$a_{3,0}(t) = -\sum_{n\geq 2}\frac{1}{(2n+1)}\{a_{1,n}(t)\}^2 \quad (61)$$

The isotropic component of the pressure impulse (i.e., the term proportional to A_0) does not influence the bubble oscillation at all. The internal pressure through terms of $O(\varepsilon^2)$ is

$$G(t) = -\varepsilon^2\sum_{n\geq 2}\frac{(n+1)A_n^2}{4(2n+1)}\{3+(4n-1)\cos 2\omega_n t\} + O(\varepsilon^3). \quad (62)$$

Comparison of this result with (59) will serve to emphasize the fact that the term of $O(\varepsilon^{3/2})$ found earlier is strictly due to interaction between the $O(\varepsilon)$ oscillation and the mean flow, and is thus zero in the absence of the flow.

Finally, it is noteworthy that the general features of the results discussed in this section for bubble oscillations caused by an impulsive change in the pressure in the presence of a uniaxial flow, are *not* altered if we consider a *step* change in pressure, i.e.,

$$A(\theta,\ t) = \sum_n A_n P_n(\zeta)\ H(t) \quad (63)$$

where H(t) is the Heaviside step function. For the step change, the leading order solution is given by

$$a_{1,n}(t,\ \tau) = \frac{1}{2}\left[\beta_{1,n}(\tau)e^{i\omega_n t} + a_{1,n}^s - \frac{A_n(n+1)}{\omega_n^2}\right] + \text{c.c.} \quad (64)$$

$$b_{1,n}(t,\ \tau) = -\frac{i\omega_n}{2(n+1)}\beta_{1,n}(\tau)e^{i\omega_n t} + \text{c.c.} \quad (65)$$

with

$$\beta_{1,n}(0) = \frac{A_n(n+1)}{\omega_n^2} \quad (66)$$

instead of (40)-(42) for the impulse. Thus, the bubble oscillates around a new steady-state shape $a_{1,n}^s - (n+1)A_n/\omega_n^2$ with an amplitude $(n+1)A_n/\omega_n^2$.

CONCLUSIONS

In this study, we consider the effects of an external flow on the dynamic response of a constant-volume bubble to changes in the ambient pressure. We find that, for an incompressible bubble, a radial mode oscillation is present only as a consequence of the constant-volume constraint, and cannot be created by the mode-mode interactions (of the deformation modes) that are incorporated into the kinematic and the dynamic boundary conditions. In this case, as shown in section 3, a monopole pressure radiation can be created at large distances *only* by an interaction between the $S_n(\theta,\ \varphi)$-mode of deformation and an external mean flow which varies like $\Phi^u \sim r^{n+2}S_{n+2}(\theta,\ \varphi)$ as $r\to\infty$. Therefore, in the absence of an external flow, the disturbance pressure caused by motion of a constant-volume bubble exhibits at most a *quadrupole* character that is associated with the $S_2(\theta,\ \varphi)$-mode of shape oscillation. Accordingly, the decay of the disturbance pressure is faster than the monopole decay for an ideal gas bubble.

We also examine small-amplitude oscillations of a constant volume bubble in response to an 'abrupt' change in the ambient pressure in the presence of a uniaxial extensional flow. The result of the asymptotic analysis for small-amplitude oscillations shows that a self-induced secularity always arises at $O(\varepsilon^2)$ due to the non-spherical steady-state shape in the straining flow. The secularity leads to the modification of oscillation frequency, which decreases as Weber number increases. For a constant-volume bubble, the square of the frequency goes to zero at $\varepsilon = 1.616$. Thus, a bubble in an inviscid straining flow would be unstable for Weber numbers larger than 1.616, which is very close to the numerical prediction of 1.38 by others. For an incompressible bubble, a radial mode of oscillation is present at $O(\varepsilon^2)$ only to *cancel* volume changes associated with the change in shape at $O(\varepsilon)$, and the disturbance flow-field does not contain a source-like disturbance at *any* order in ε. In addition, the frequency of this radial mode for a constant-volume bubble will be quite different from the natural frequency ω_0 for radial oscillations of an ideal gas bubble.

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